What is a perfectoid space?

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/-- A perfectoid ring, following Fontaine Sem Bourb-/
class perfectoid_ring (R : Type*) extends Tate_ring R :=
(complete : @is_complete_hausdorff R (topological_add_group.to_uniform_space R))
(uniform  : is_uniform R)
(ramified : ∃ ϖ : units R, (is_pseudo_uniformizer ϖ) ∧ ((ϖ^p : R°) | p))
(Frob     : ∀ a : R°, ∃ b : R°, (p : R°) | (b^p - a))

class perfectoid_space (X : Type*) extends adic_space X :=
(perfectoid_cover : ∀ x : X, ∃ (U : opens X) (A : Huber_pair) [perfectoid_ring A],
  (x ∈ U) ∧ is_preadic_space_equiv U (Spa A))
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OK so it doesn't quite work, and there are a few sorries, but ultimately a perfectoid space is a complicated algebraic structure ("a topological space plus a sheaf of complete topological rings which locally looks like the adic spectrum of a perfectoid ring").

But what is the point of them?

Given a geometric object (for example a topological space), one can consider some appropriate class of functions on this object (for example the continuous real-valued functions), and using pointwise addition and multiplication these functions form a commutative ring (all rings are commutative in this talk). More generally, one can consider the construction which sends an open set in this space to the ring of continuous functions on this open set, and this data is called a *presheaf of rings*. In fact this presheaf satisfies one extra axiom called the sheaf axiom, and the corresponding *ringed space*, i.e., a topological space equipped with a sheaf of rings, is perhaps one good working definition of a "geometric object".

Grothendieck turned this idea on its head. He observed that given just a commutative ring R, one can actually build a topological space $\mathrm{Spec}(R)$ equipped with a sheaf of rings, and the functions on this space (i.e. the result of evaluating the sheaf on $\mathrm{Spec}(R)$ itself) is R again. Grothendieck called such spaces *affine schemes*.

Grothendieck defined a *scheme* to be a ringed space which can be covered by open sets which are isomorphic to affine schemes. In other words, a scheme is a geometric object which can be analysed locally using commutative ring theory.

Much of the original application of Grothendieck's work (new cohomology theories, a proof of the Weil conjectures and so on) used *Noetherian* schemes, which are schemes with a crucial finiteness property (the corresponding rings have the property that all ideals are finitely generated).

The Langlands Programme.

The Langlands programme is a web of conjectures and ideas relating "algebra" (the representation theory of Galois groups) to "analysis" (spaces of harmonic forms on certain locally symmetric spaces). For those that know some of this kind of mathematics, an elliptic curve over the rationals gives gives a representation of a Galois group via its Tate module, and a modular form is an example of a harmonic function on the upper half plane; the Shimura–Taniyama conjecture relates these two objects, and its proof in many cases by Wiles enabled him to deduce Fermat's Last Theorem.

These spaces coming from analysis are infinite-dimensional, but the best-understood parts are those related to the cohomology of the symmetric spaces in question (for example, modular forms contribute to the cohomology of modular curves). However, although each modular curve can be regarded as a Noetherian scheme, the limit of all of them is no longer Noetherian, making it hard to analyse.

Adic spaces.

Around 1990, Roland Huber introduced a new, more analytic class of objects, called adic spaces. These are topological spaces equipped with sheaves of topological rings, making them more suited to analysis. If R is a certain kind of topological ring, then Huber considers a variant of Spec(R) called Spa(R), a richer topological space called an *affinoid adic space*. A space which is locally affinoid is called an *adic space*. Like schemes, adic spaces may or may not be Noetherian, and Huber's main results on these spaces were restricted to Noetherian adic spaces.

Perfectoid spaces.

Scholze came up with another algebraic property of topological rings, and a ring with this property is called a *perfectoid ring*. Perfectoid rings are very rarely Noetherian – the extra algebraic axiom Scholze added involved assuming the existence of pth roots of essentially every element of the ring, where p is a fixed prime number, which typically forces these rings to be non-Noetherian. However Scholze managed to isolate a different kind of useful property which controlled how these rings behaved, called "tilting", and initially used this property to prove new cases of the monodromy-weight conjecture in algebraic geometry, by reducing a question in characteristic 0 to a question in characteristic p where it had already been resolved.

What was surprising at the time (at least to me) was that this notion actually had uses well beyond its original intention. One crucial observation, due to Scholze and Weinstein, was that these limits of locally symmetric spaces showing up in the Langlands philosophy (for example the limit of the tower of modular curves $\cdots \to X(p^n) \to \cdots \to X(p^2) \to X(p) \to X(1)$) could be given the structure of a perfectoid space. The cohomology of this perfectoid space was a crucial new ingredient in the relatively new area of the p-adic Langlands philosophy, and quickly new results were appearing, not just about the p-adic Langlands philosophy but the classical Langlands philosophy. These applications were one of the main reasons that Scholze was awarded the Fields Medal last August.

Scholze's *tilting correspondence* is a theorem about the category of sheaves on a perfectoid space. A huge amount of arithmetic geometry and arithmetic goes into Scholze's proof (for example the Gabber–Romero results on "almost ring theory" play a key role, and this paper is over 1600 pages long, and whilst these results would be a joy to formalise in Lean because the authors are so careful, it is clear that we will not be seeing a proof of the tilting correspondence in Lean any time soon. However there is nothing stopping mathematicians from formalising the definition of a perfectoid space and the *statement* of the tilting correspondence.

Patrick Massot, Johan Commelin and I are working on the formalisation of the definition of a perfectoid space in Lean.

Status of the project.

Given an appropriate topological ring R, we have defined the topological space Spa(R), but we still need to define the sheaf of rings (actually we only need to define the presheaf of rings). Once this is done, we are finished, because we have the perfectoid ring predicate and getting from an affinoid perfectoid space to a perfectoid space is easy.
This basically reduces the work needed to a lot of commutative ring theory, and we have done a bunch of it, but there is more left. Patrick has done a lot of work defining the completion of a topological ring and proving the universal property. Johan has been using Scott Morrison's category theory mathlib contributions to set up sheaves of rings within this framework. I will be actively working on the repo over the next few months and we hope to finish it soon (I know I always say this, but I see no technical obstructions to finishing the project, it is just an issue of the authors being busy).