Ring completions in the perfectoid project

Patrick Massot (Orsay)

joint work with

Kevin Buzzard (IC London), Johan Commelin (Freiburg), ...

Lean Together
January 8th 2019

Lean Forward

The perfectoid project: history

Kevin Buzzard @kbuzzard Dec 01 2017 23:51

I would be interested in your experiences in talking to people about your interest in this stuff

A lot of people I've talked to simply have no interest or just can't see the point

Find some strong algebraists

and ask them what they think about the computer proof of the odd order theorem

Patrick Massot @PatrickMassot Dec 01 2017 23:52

I'm affraid strong algebraists in Orsay don't care about the odd order theorem, computer or not

Kevin Buzzard @kbuzzard Dec 01 2017 23:52

heh

Patrick Massot @PatrickMassot Dec 01 2017 23:53

Let me try with Laumon and Fontaine for a laugh

Clearly people in Orsay will want to see perfectoid spaces in Lean before being impressed

Kevin Buzzard @kbuzzard Dec 01 2017 23:56

funny you mention that

I was talking to Mario earlier about implementing them

The perfectoid project: the team

  

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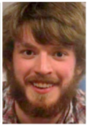

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Why perfectoid spaces?

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> *The key issue for me is finding the right definitions, finding the right notions that really capture the essence of some mathematical phenomenon. I often have some vague vision of what I want to understand, but I'm often missing the words to really say that.*

> *The essential difficulty in writing “Étale cohomology of diamonds” was (by far) not giving the proofs, but finding the definitions. But even beyond mere language, we perceive mathematical nature through the lenses given by definitions, and it is critical that the definitions put the essential points into focus.*

Composing limits

Limits

Live demo: composing limits.

Filters: definition

Definition

A filter on a type X is a set $\mathcal{F}$ of subsets of X such that

- $X \in \mathcal{F}$
- $(U \in \mathcal{F}$ and $U \subset V) \Rightarrow V \in \mathcal{F}$
- $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$

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Example

- X topological space, $x \in X$, $\mathcal{N}_x = \{$neighborhoods of x$\}$
- $X = \mathbb{N}$, $\mathcal{N}_\infty = \{$complements of finite subsets$\}$
- $X = \mathbb{R}$, $\mathcal{N}_{+\infty} = \{U$ containing some $[A, +\infty)\}$
- $X = \mathbb{R}$, $a \in X$, $\mathcal{N}_{a^+} = \{U$ containing some $[x, x+\varepsilon), \varepsilon > 0\}$

Filters: limits and composition

Definition

$f : X \to Y$ tends to $\mathcal{G} \in \mathrm{Filter}(Y)$ along $\mathcal{F} \in \mathrm{Filter}(X)$ if

$$\forall V \in G, \quad f^{-1}(V) \in \mathcal{F}.$$

Limits compose (Live demo)

Filters: order, push-forward and limits

Partial order: $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{G} \subset \mathcal{F}$.

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$f : X \to Y$, $\mathcal{F} \in \mathrm{Filter}(X) \rightsquigarrow f_*\mathcal{F} \in \mathrm{Filter}(Y)$

$$f_*\mathcal{F} := \{V \subset Y \; ; \; f^{-1}(V) \in \mathcal{F}\}.$$

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Composition of limits

Lemma

- f_* *is non-increasing:* $\mathcal{F}_1 \leq \mathcal{F}_2 \Rightarrow f_* \mathcal{F}_1 \leq f_* \mathcal{F}_2$
- $(g \circ f)_* = g_* \circ f_*$

Corollary

Limits compose.

(Lean)

Filters: pull-back and Galois connection

$f : X \to Y$, $\mathcal{G} \in \mathrm{Filter}(Y) \rightsquigarrow f^*\mathcal{G} \in \mathrm{Filter}(X)$

$$f^*\mathcal{G} := \{U \subset X \mid \exists V \in \mathcal{G}, f^{-1}(V) \subset U\}.$$

If f is injective then $f^*\mathcal{G} := \{f^{-1}(V)\}_{V \in \mathcal{G}}$.

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Example: $X \subset Y$ equipped with subspace topology, $\iota : X \hookrightarrow Y$, $\mathcal{N}_x^X = \iota^*\mathcal{N}_x^Y$. It was secretly used in our filter examples slide.

This is *not* inverse to push-forward, but it's also non-increasing and:

$$f_*\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F} \leq f^*\mathcal{G}.$$

Topological rings and uniform spaces

Another view on $\mathbb{Z}_p$

From Rob's talk: $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$. Direct definition?

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Using $\pi_n : \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$

$$\begin{aligned} \mathbb{Z}_p &= \varprojlim \mathbb{Z}/p^n \\ &= \left\{ (a_n)_{n\geq 1} \in \prod_{n\geq 1} \mathbb{Z}/p^n \;\middle|\; \pi_n(a_{n+1}) = a_n \right\}. \end{aligned}$$

Link with the completion idea?

I-adic topology

In $\mathbb{Z}$, define $\mathcal{N}_0 := \{U \mid \exists n, p^n\mathbb{Z} \subset U\}$

and, for any $a \in \mathbb{Z}$, $\mathcal{N}_a = (b \mapsto a + b)_* \mathcal{N}_0$

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It can fail to be a metric topology (non-Hausdorff).

Cauchy sequences?

Problem: a topology is not enough data to talk about completions.
Remember a sequence (u_n) is Cauchy if

$$\forall \varepsilon > 0,\ \exists N,\ \forall m, n \geq N, \quad |u_m - u_n| \leq \varepsilon.$$

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Generalize to either

$$\forall \varepsilon > 0,\ \exists N,\ \forall m, n \geq N, \quad d(u_n, u_m) \leq \varepsilon$$

or

$$\forall U \in \mathcal{N}_0,\ \exists N,\ \forall m, n \geq N, \quad u_m - u_n \in U.$$

Uniform spaces

Definition

A uniform structure on a set X is a filter $\mathcal{U}$ on $X \times X$ such that:

- every $V \in \mathcal{U}$ contains the diagonal
- $((x,y) \mapsto (y,x))_* \mathcal{U} \leq \mathcal{U}$
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(X, d) metric space $\rightsquigarrow V \in \mathcal{U} \Leftrightarrow \exists \varepsilon > 0, \{d(x, x') < \varepsilon\} \subset V$

$(G, +)$ additive topological group $\rightsquigarrow \mathcal{U} = (-)^* \mathcal{N}_0$.

Uniform continuity

Uniform continuity

Recall $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if:

$$\forall \varepsilon \;\; \exists \eta \;\; \forall x, y \quad |x - y| \leq \eta \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

Definition

A function $f : X \to Y$ between uniform spaces is uniformly continuous if

$$\forall V \in \mathcal{U}_Y, \exists U \in \mathcal{U}_X, (x, x') \in U \Rightarrow (f(x), f(x')) \in V.$$

Lemma

Uniform continuous implies continuous

(Lean)

Uniform continuous implies continuous

Alternative definition: $f : X \to Y$ is uniformly continuous if $(f \times f)_* \mathcal{U}_X \leq \mathcal{U}_Y$ or, equivalently, $\mathcal{U}_X \leq (f \times f)^* \mathcal{U}_Y$

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Proof of continuity: Fix any $x \in X$, remember $\mathcal{N}_x = \iota_x^* \mathcal{U}_X$, and compute, still using $(f \times f) \circ \iota_x = \iota_{f(x)} \circ f$:

$$\begin{aligned}
\mathcal{N}_x &= \iota_x^* \mathcal{U}_X \\
&\leq \iota_x^* (f \times f)^* \mathcal{U}_X \\
&= \iota_x^* (f \times f)^* \mathcal{U}_X \\
&= f^* \iota_{f(x)}^* \mathcal{U}_X \\
&= f^* \mathcal{N}_{f(x)}
\end{aligned}$$

So $\mathcal{N}_x \leq f^* \mathcal{N}_{f(x)}$, hence $f_* \mathcal{N}_x \leq \mathcal{N}_{f(x)}$.

Completions

Completion functor

One can define completeness for uniform spaces (skipped here).

We want, for each uniform space X, a complete one $\hat{X}$, and

$i_X : X \to \hat{X}$ such that $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ \hat{X} & \underset{\exists! \hat{f}}{\dashrightarrow} & \hat{Y} \end{array}$ commutes

With $\hat{X}$ "minimal", $\widehat{f \circ g} = \hat{f} \circ \hat{g}$, and $\widehat{\mathrm{id}_X} = \mathrm{id}_{\hat{X}}$.

Minimal?

$(\hat{X}, i_X)$ minimal means that, for every map f into a *complete* Z

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle i_X}\big\downarrow & \nearrow_{\exists! \tilde{f}} & \\ \hat{X} & & \end{array}$$

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This $\tilde{\ }$ allows to construct $\widehat{\ }$ on maps:

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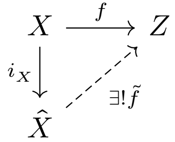

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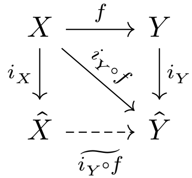

The case of groups

G topological group (eg. additive structure on a topological ring)
We want

- topological group structure on $\hat{G}$
- i_G a group morphism
- $f : G \to K$ (continuous) group morphism into *complete* group K implies $\tilde{f}$ group morphism

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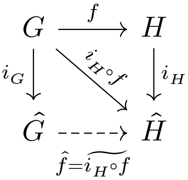
Hence $\hat{f}$ is a group morphism since:



(Switch to Lean)

The issue

We have:

$$\begin{array}{ccccc} & (G,+,\mathcal{U}_{+,\mathcal{T}}) \rightsquigarrow (\hat{G},\hat{+},\widehat{\mathcal{U}_{+,\mathcal{T}}}) & & & (\hat{G},\hat{+},\mathcal{U}_{\hat{+},\widehat{\mathcal{T}}}) \\ \nearrow & & \searrow & \nearrow & \\ (G,+,\mathcal{T}) & & & (\hat{G},\hat{+},\widehat{\mathcal{T}}) & \end{array}$$

But $\widehat{\mathcal{U}_{+,\mathcal{T}}} = \mathcal{U}_{\hat{+},\widehat{\mathcal{T}}}$ is not obvious in any way.

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$$(G, +, \mathcal{T}) \rightsquigarrow (G, +, \mathcal{U}_{+,\mathcal{T}}) \rightsquigarrow (\hat{G}, \hat{+}, \widehat{\mathcal{U}_{+,\mathcal{T}}}) \rightsquigarrow (\hat{G}, \hat{+}, \widehat{\mathcal{T}}) \rightsquigarrow (\hat{G}, \hat{+}, \mathcal{U}_{\hat{+},\widehat{\mathcal{T}}})$$

But $\widehat{\mathcal{U}_{+,\mathcal{T}}} = \mathcal{U}_{\hat{+},\widehat{\mathcal{T}}}$ is not obvious in any way.

Bourbaki's solution:

- if X is any random uniform space, its completion is denoted by $\hat{X}$, functorial properties of $\hat{X}$ and i_X are proved.
- if G is a topological group, prove there is a topological group which is complete, has a morphism from G etc. Denote it by $\hat{G}$.

A better setup

Definition

A uniform additive group is $(G, +, \mathcal{U})$ such that subtraction is uniformly continuous.

Lemma

- $\forall (G, +, \mathcal{U})$ *uniform add group,* $\mathcal{U} = (-)^* \mathcal{N}_0$.
- $\forall (G, +, \mathcal{T})$ *commutative,* $(G, +, (-)^* \mathcal{N}_0)$ *is a uniform add group.*
- $\forall (G, +, \mathcal{U})$ *uniform add group, there exists* $\hat{+}$ *on* $\hat{G}$ *such that* $(\hat{G}, \hat{+}, \widehat{\mathcal{U}})$ *is a uniform add group.*