Formalizing the *p*-adic Numbers in Lean

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Motivation

A project at the VU: formalize modern results in number theory, in Lean.

- Develop comprehensive libraries that will help with many results.
- Target “research areas”/collections of moderate difficulty results, instead of single challenge theorems.
- Work on the system and automation alongside the formalizing.

Formalizing number theory

Number theory starts as “the study of $\mathbb{Z}$” but quickly goes beyond this.

We need libraries for:

- computations on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$: divisibility, modularity, factoring, arithmetic, inequalities, …
- less familiar “number” structures, such as number fields, the p-adic numbers, …
- univariate and multivatiate polynomials, and related algebra and geometry
- special functions: Dirichlet series, modular forms, …

Completions

The rational numbers $\mathbb{Q}$ are incomplete.

The sets

$$\left\{x \in \mathbb{Q} \,\middle|\, x^2 < 2\right\}$$

$$\left\{x \in \mathbb{Q} \,\middle|\, x^2 > 2\right\}$$

partition $\mathbb{Q}$, but both are open.

Alternatively: the sequence of rationals

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, 1.41421356, \ldots$$

does not converge to a rational.

Completing $\mathbb{Q}$

Definition.

A sequence $s : \mathbb{N} \to \mathbb{Q}$ is Cauchy if for every positive $\epsilon \in \mathbb{Q}$, there exists a number N such that for all $k \geq N$, $|s_N - s_k| < \epsilon$.

Intuition: a sequence is Cauchy if its entries eventually become arbitrarily close.

Definition.

Two sequences s and t are equivalent, written $s \sim t$, if for every positive $\epsilon \in \mathbb{Q}$, there exists an N such that for all $k \geq N$, $|s_k - t_k| < \epsilon$.

Intuition: two sequences are equivalent if they eventually become arbitrarily close to each other.

Definition.

A sequence $s : \mathbb{N} \to \mathbb{Q}$ is Cauchy if for every positive $\epsilon \in \mathbb{Q}$, there exists a number N such that for all $k \geq N$, $|s_N - s_k| < \epsilon$.

Intuition: a sequence is Cauchy if its entries eventually become arbitrarily close.

Claim.

The sequence

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, 1.41421356, \ldots$$

is Cauchy. Why?

The real numbers

We think of the real numbers $\mathbb{R}$ as $\mathbb{Q}$ plus points in the "gaps."

Cauchy sequences identify these points.

Definition?

The set of real numbers $\mathbb{R}$ is the set $\left\{ s : \mathbb{N} \to \mathbb{Q} \;\middle|\; s \text{ is Cauchy} \right\}$.

The real numbers

We think of the real numbers $\mathbb{R}$ as $\mathbb{Q}$ plus points in the "gaps."

Cauchy sequences identify these points.

Problem!

Some Cauchy sequences identify the same "points."

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, 1.41421356, \ldots$$

$$2, 1.5, 1.42, 1.415, 1.4143, 1.41422, 1.414214, 1.4142136, 1.41421357, \ldots$$

Quotients

Definition.

A binary relation is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition.

Let S be a set, $\sim$ an equivalence relation on S, and $a \in S$. The equivalence class of a with respect to $\sim$, denoted $[\![a]\!]$, is the set $\left\{x \in S \,\middle|\, a \sim x\right\}$. The quotient of S with respect to $\sim$, denoted $S/\sim$, is the set $\left\{[\![a]\!] \,\middle|\, a \in S\right\}$.

Definition.

Two sequences s and t are equivalent, written $s \sim t$, if for every positive $\epsilon \in \mathbb{Q}$, there exists an N such that for all $k \geq N$, $|s_k - t_k| < \epsilon$.

Claim.

The relation $\sim$ is an equivalence relation.

Definition.

The set of real numbers $\mathbb{R}$ is the quotient of the set of rational Cauchy sequences, with respect to $\sim$. We call this the completion of $\mathbb{Q}$.

We define addition of sequences in the obvious way.

Claim.

If $r_1 \sim r_2$ and $s_1 \sim s_2$ then $r_1 + s_1 \sim r_2 + s_2$.

This lets us define addition on $\mathbb{R}$: $[\![r]\!] + [\![s]\!] = [\![r + s]\!]$.

Similarly for multiplication, etc.

Question:

In the construction of $\mathbb{R}$, what was hardcoded? What can we abstract?

General completions

We can generalize the measure of distance.

Definition.

A sequence $s : \mathbb{N} \to \mathbb{Q}$ is Cauchy if for every positive $\epsilon \in \mathbb{Q}$, there exists a number N such that for all $k \geq N$, $|s_N - s_k| < \epsilon$.

General completions

We can generalize the measure of distance.

Definition.

A sequence $s : \mathbb{N} \to \mathbb{Q}$ is Cauchy with respect to an absolute value abs if for every positive $\epsilon \in \mathbb{Q}$, there exists a number N such that for all $k \geq N$, $\text{abs}(s_N - s_k) < \epsilon$.

Definition.

A function abs on $\mathbb{Q}$ is a (generic) absolute value if it is

- positive-definite: $\text{abs}(0) = 0$ and $\text{abs}(k) > 0$ otherwise
- subadditive: $\text{abs}(x + y) \leq \text{abs}(x) + \text{abs}(y)$
- multiplicative: $\text{abs}(x \cdot y) = \text{abs}(x) \cdot \text{abs}(y)$

General completions

We can also generalize the base type from $\mathbb{Q}$ to any metric space.

But we'll focus on $\mathbb{Q}$ for today.

Absolute values on $\mathbb{Q}$

Definition.

A function abs on $\mathbb{Q}$ is a (generic) absolute value if it is

- positive-definite: $\mathrm{abs}(0) = 0$ and $\mathrm{abs}(k) > 0$ otherwise
- subadditive: $\mathrm{abs}(x + y) \leq \mathrm{abs}(x) + \mathrm{abs}(y)$
- multiplicative: $\mathrm{abs}(x \cdot y) = \mathrm{abs}(x) \cdot \mathrm{abs}(y)$

Example.

The trivial absolute value on $\mathbb{Q}$ is given by

$$|x|_0 = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

The p-adic norm

The p-adic valuation

Fix a natural number $p > 1$.

Definition.

The *p-adic valuation* $\nu_p : \mathbb{Z} \to \mathbb{N}$ is defined by

$$\nu_p(z) = \max \left\{ n \in \mathbb{N} \,\middle|\, p^n \mid z \right\}$$

with $\nu_p(0) = \infty$ (or 0, we don't care for now).

This extends to $\nu_p : \mathbb{Q} \to \mathbb{Z}$ by setting

$$\nu_p(q/r) = \nu_p(q) - \nu_p(r)$$

when q and r are coprime.

The p-adic norm

$$\nu_p(z) = \max\left\{n \in \mathbb{N} \,\middle|\, p^n \mid z\right\}$$

$$\nu_p(q/r) = \nu_p(q) - \nu_p(r)$$

Definition.

The p-adic norm $|\cdot|_p : \mathbb{Q} \to \mathbb{Q}$ is defined by

$$|x|_p = \begin{cases} 0 & x = 0 \\ \frac{1}{p^{\nu_p(x)}} & x \neq 0 \end{cases}$$

The p-adic norm

Examples.

x	$v_3(x)$	$\|x\|_3$
1	0	1
3	1	$\frac{1}{3}$
6	1	$\frac{1}{3}$
18	2	$\frac{1}{9}$
$\frac{1}{3}$	-1	3
118098	10	$\frac{1}{59049}$
118099	0	1

The p-adic norm

When p is prime, the p-adic norm is an absolute value on $\mathbb{Q}$.

So we can complete $\mathbb{Q}$ with respect to $|\cdot|_p$.

The result: the p-adic numbers $\mathbb{Q}_p$.

The p-adic numbers

The p-adic numbers

A real number in base 10 is

$$\pm \sum_{i=-\infty}^{k} a_i \cdot 10^i$$

where k is a (possibly negative) integer and each $a_i \in \{0, 1, \ldots, 9\}$.

A p-adic number in base p is

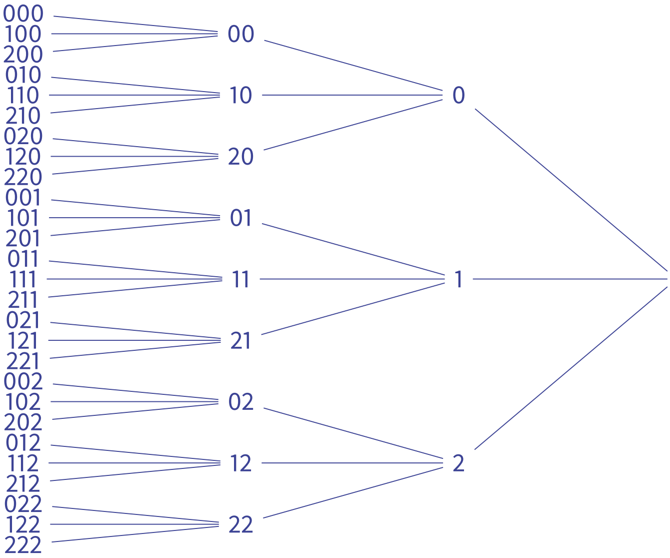
$$\sum_{i=k}^{\infty} a_i \cdot p^i$$

where k is a (possibly negative) integer and each $a_i \in \{0, 1, \ldots, p-1\}$.

Arithmetic in $\mathbb{Q}_5$

$$\begin{array}{r} {\scriptstyle 1\,1\,1\,1\,1\,1\,1\;\,} \\ \ldots 44444444 \\ +\quad\quad\quad\quad 1 \\ \hline 0 \end{array} \qquad \begin{array}{r} {\scriptstyle 1\,2\,1\,2\,1\,2\,1\;\,} \\ \ldots 31313132 \\ \times\quad\quad\quad\quad 3 \\ \hline 1 \end{array} \qquad \begin{array}{r} {\scriptstyle 1\,1\,1\,1\,1\,1\,1\;\,} \\ \ldots 31313132 \\ +\ \ldots 44444444 \\ \hline \ldots 31313131 \end{array}$$

The p-adic numbers



Properties of the p-adic norm

The p-adic norm on $\mathbb{Q}$ lifts to $\mathbb{Q}_p$.

(Reason: for any Cauchy sequence $s : \mathbb{N} \to \mathbb{Q}$, $|s_i|_p$ is eventually constant.)

Theorem.

The p-adic norms on $\mathbb{Q}$ and $\mathbb{Q}_p$ are nonarchimedean. That is, for all x and y,

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

This simplifies many things in the study of $\mathbb{Q}_p$.

The p-adic integers

A consequence of the nonarchimedean property: if $|x|_p \leq 1$ and $|y_p| \leq 1$, then $|x+y|_p \leq 1$.

Definition.

The p-adic integers $\mathbb{Z}_p$ are the set

$$\left\{ z \in \mathbb{Q}_p \;\middle|\; |z|_p \leq 1 \right\}.$$

This set forms a ring.

Hensel's lemma

Let $\mathbb{Z}_p[X]$ denote the set of polynomials with coefficients in $\mathbb{Z}_p$.

Theorem.

Suppose that $f(X) \in \mathbb{Z}_p[X]$ and $a \in \mathbb{Z}_p$ satisfy $|f(a)|_p < |f'(a)|_p^2$. There exists a unique $z \in \mathbb{Z}_p$ such that $f(z) = 0$ and $|z - a|_p < |f'(a)|_p$.